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The finite and semi-infinite tilted, flat but rounded punch

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Abstract

The problem of a tilted flat punch having a rounded edge, and with a sufficiently large angle of tilt for contact to be lost along the flat face, is considered. A complete solution to the contact problem, within the context of an elastic half-plane formulation, is derived, including the effects of a shearing force either sufficient or insufficient to cause sliding. The solution is then modified by making the punch semi-infinite in extent, so as to render it effective as an asymptote useful in both quantifying fretting damage, and in improving the precision of approximate numerical solutions. The asymptote is then applied to an example problem.

Keywords: Tilted flat and rounded punch; Contact; Partial slip; Asymptotic forms

1. Introduction

This paper is concerned with the idealised contact problem shown in Fig. 1. It consists of a punch having a flat face, but with a radius, R, present at the ends. The normal load, P, is applied at a distance s' from the flat-rounded transition point (and with the sign sense indicated), so that the right-hand side of the punch lifts out of contact lifts out of contact. This means that both edges of the contact itself are 'incomplete' in character, and the extent of the contact needs to be quantified. The problem has been studied for several reasons: first, the geometry itself is very simple in nature. Hence, it yields a closed form solution, including the contact pressure, $p(x)$, angle of tilt, α, and extent of contact $[d,l]$. A shearing force, Q, applied in the plane of the contact is then added. Both sliding and partial slip cases (the latter when $|Q| < fP$, where f is the coefficient of friction) is then applied, and the resulting stick-slip regime found, together with the interior state of stress. Secondly, the problem is of help in understanding the behaviour

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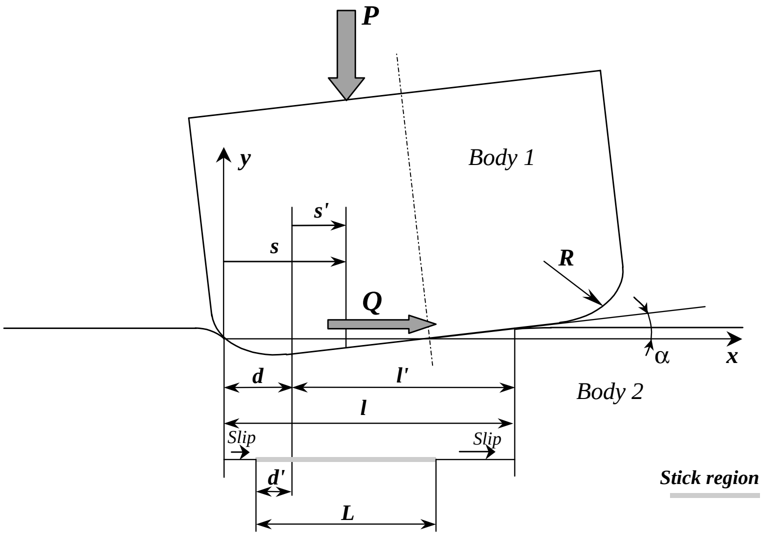

Fig. 1. Flat-rounded tilted punch, and inset, the partial slip regime.

of dovetail roots of gas turbine fan blades, where recently the 'flat and rounded punch', symmetrically loaded, has often been used to represent the dovetail flank contact. Here, we have added the means of looking at the effects of tilt, imposed by vibrational loading. Lastly, the solution provides a vehicle for investigating the possibility of generating a semi-infinite flat and rounded, but also tilted asymptotic contact solution, and this is discussed, together with related issues concerning first-order asymptotes to this solution.

2. Formulation

Fig. 1 shows the problem under consideration. The first question which arises is this; if half-plane theory is to be used in the analysis, which combinations of elastic constants may correctly be permitted in the solution? This is not an easy question to answer, but the solution to be derived is certainly correct if the punch is rigid, and the half-plane incompressible. With this assumption the elastic composite compliance, A, is given by

$$A = \frac{1}{\mu}, \tag{1}$$

where μ is the modulus of rigidity of the half-plane. It is arguable that, providing that the length of punch remaining in contact is small ($l \ll R$), the solution may also be appropriate when both components have a finite elasticity, and each body capable of being represented by a half-plane. In this case it is still necessary to ensure that there is no coupling between the direct and shear components of traction, so that either the components must be elastically similar, or the interface must be perfectly lubricated ($f = 0$). With this provison, in the most general case, we have

$$A = \frac{\kappa_1 + 1}{2\mu_1} + \frac{\kappa_2 + 1}{2\mu_2} \tag{2}$$

where κ_i is Kolosov's constant ($=3-4\nu_i$ where ν_i is Poisson's ratio for body i), and μ_i the corresponding modulus of rigidity.

The punch is pressed into an elastic half-plane. A portion of the rounded section (from $x=0$ to $x=d$ say) together with a finite portion (from $x=d$ to $x=l$ say) of the flat section of the profile indents the half-plane. The contacting rounded portion is represented by the usual Hertz parabolic approximation to a circle from $x=0$ to $x=d$ so the relative surface normal displacement is

$$v(x)=\begin{cases}\Delta+\alpha x-\dfrac{(x-d)^2}{2R}, & 0\leqslant x\leqslant d\\ \Delta+\alpha x, & d\leqslant x\leqslant l\end{cases} \tag{3}$$

where Δ is a constant measuring the absolute indentation. If $\alpha=0$ then the punch is 'flat', but in any case α must also be sufficiently small for small strain conventional elasticity theory to apply, which, as with all elasticity problems of this class, also implies a maximum value for the load, P.

The connection between the displacement and the surface normal pressure, $p(x)$, is

$$\frac{1}{A}\frac{\partial v}{\partial x}=-\frac{1}{\pi}\int_0^l\frac{p(t)}{t-x}\mathrm{d}t \tag{4}$$

Differentiating (3) with respect to x and substituting into Eq. (4) gives

$$\frac{1}{\pi}\int_0^l\frac{p(t)}{t-x}\mathrm{d}t=\frac{g(x)}{A}, \tag{5}$$

where

$$g(x)=-\begin{cases}\alpha+\beta\left(1-\dfrac{x}{d}\right), & 0\leqslant x\leqslant d\\ \alpha, & d\leqslant x\leqslant l\end{cases},\quad \beta=\frac{d}{R} \tag{6}$$

The solution is found in Appendix A, and is given by

$$p(x)=\frac{\beta}{A\pi}\left\{\frac{x-d}{d}\ln\left|\frac{\sqrt{\dfrac{x}{l-x}}+\sqrt{\dfrac{d}{l-d}}}{\sqrt{\dfrac{x}{l-x}}-\sqrt{\dfrac{d}{l-d}}}\right|-\frac{\sqrt{x(l-x)}}{d}\cos^{-1}\left(1-\frac{2d}{l}\right)\right\} \tag{7}$$

subject to a consistency condition

$$\frac{\alpha}{\beta}=\frac{1}{\pi}\left\{\frac{c\cos^{-1}c-\sqrt{1-c^2}}{1-c}\right\} \tag{8}$$

$$c=1-2\left(\frac{d}{l}\right). \tag{9}$$

In addition to this side condition, overall equilibrium must be satisfied, which provides the following two requirements:

$$P=\int_0^l p(x)\mathrm{d}x \tag{10}$$

$$Ps = \int_0^l p(x) x \, dx \tag{11}$$

The first of these gives rise to the following relation (see Appendix A.1)

$$P = \frac{l\beta}{4A(1-c)} \left[\cos^{-1} c - c\sqrt{1-c^2} \right] \tag{12}$$

whilst the second gives (see Appendix A.2)

$$Ps = \frac{\beta l^2}{24A} \left[\frac{(1-c^2)^{3/2} + 3c\sqrt{1-c^2} - 3\cos^{-1} c}{1-c} \right] \tag{13}$$

3. Results

3.1. Contact law and contact pressure

The general characteristics of the form of the solution depend on the two dimensionless quantities (PA/R) and (s/R), the first defining the magnitude of the load, the second its point of application. These specify, using the three auxiliary equations developed above, the size of the contact $(d/R), (l/R)$, together with the angle of tilt, α . It does not seem possible to solve explicitly for these quantities, but the values of $(d/R), (l/R)$ may be found by choosing particular values of (d/l) and noting what values of the independent variables are implied. In making use of the results, attention is drawn to the fact that there is only one fixed reference position in the problem, and that is the transition point from the flat to the rounded part of the indenter face. Thus, although it is convenient to use the edge of the contact as the origin in deducing the solution, it makes sense, when considering the output from the analysis, to refer all distances to the flat/curve transition. Thus, the point of application of the load is now specified as, s'/R ($=s/R - d/R$, Fig. 1).

Fig. 2 displays, in various forms, the contact law for the problem. First, Fig. 2(a) gives the ratio d/l as a function of the load position (s'/R) and dimensionless load (PA/R): this quantity is relatively easy to deduce first. When once this has been done it is then possible to produce the contact law in the required form, and the results of this are shown in Fig. 2(b). The first point to make is that the dimensionless load has been capped at 1.0 because a larger value would compromise various aspects of the half-plane idealisation. For example, the curved part of the profile is represented as a parabola whereas it is, in fact, a part-circle, and this approximation would certainly be invalid if (d/R) exceeded, say, 0.5. Also, by this stage the amount of material beyond the contact within the punch is no longer sufficient for it to be thought of as a half-plane, and so the solution should, in this regime, be thought of as applying solely to the 'rigid indenter' case. Secondly, the distance l'/R gives the extent of the contact along the flat face. Trivially, the solution as described applies only when the actual extent of the flat face of the punch is at least as great as this. More importantly, as $l'/R \to 0$ we should recover the Hertz solution. When this condition is achieved we would expect the solution to be inherently symmetrical, as the straight portion of the boundary has no practical relevance. Thus, the value of s should be $d/2$. Further, the contact law for a conventional Hertzian contact may be written as (Hills et al., 1993):

$$\frac{PA}{R} = \frac{\pi}{2}\left(\frac{a}{R}\right)^2$$

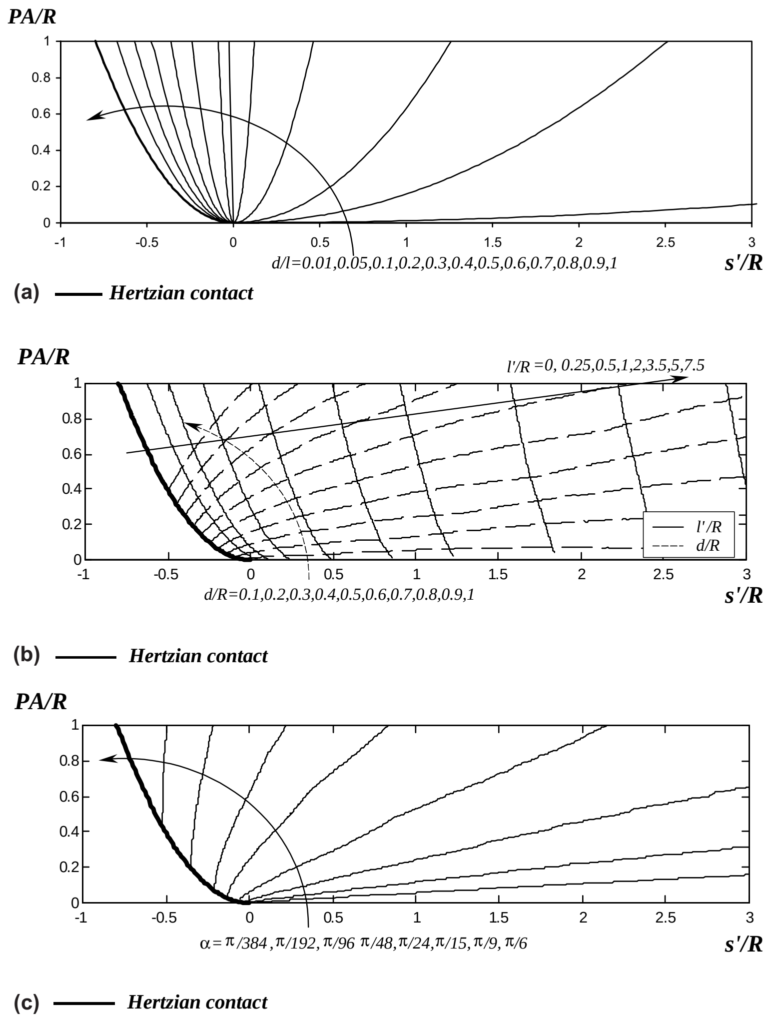

Fig. 2. (a) Ratio d/l as a function of load and its position; (b) explicit size of contact as a function of load and its position and (c) angle of tilt, α, as a function of load and its position.

where a is the contact half width. In the present nomenclature, this would mean that $d = 2a$, and hence

$$\frac{PA}{R} = \frac{\pi d^2}{8R^2} = \frac{\pi s'^2}{2R^2}$$

and this line is included in Fig. 2, indicating that the solution developed correctly goes to the Hertzian limit. The final part of the contact law is the inclination or attitude the punch adopts, α, and this is plotted separately in Fig. 2(c). Thus, if s is large the contact becomes almost 'flat', whilst as $s \to d/2$ from above $\alpha \to \pi/4$.

There are many possible ways in which the general form of the pressure distribution may be displayed. The most sensible choice seems to be to normalise the contact pressure with respect to the mean load (P/l), but to leave the dimensionless ratio (d/l) as the quantity characterising the form of the distribution, (even though both d and l are clearly dependent variables), as this best portrays the range of pressure distributions which arise. With this choice of normalisation the contact pressure becomes

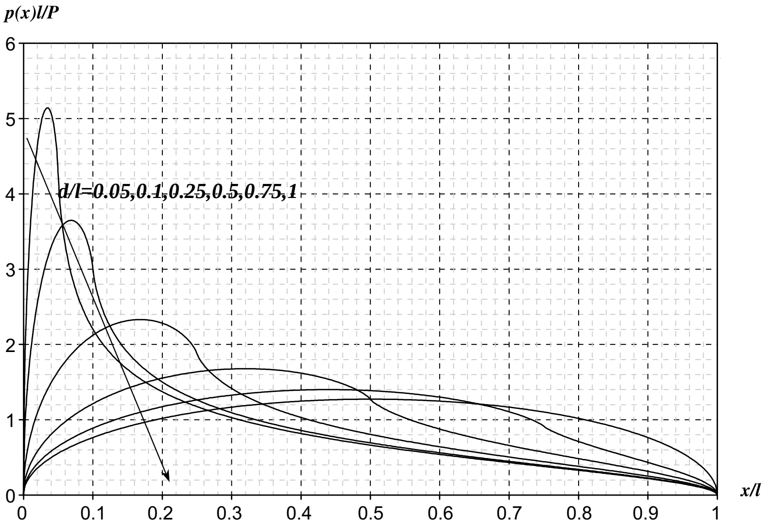

Fig. 3. Pressure distributions.

$$\frac{p(x)l}{P} = \left\{ \frac{4(1-c)}{\left[\cos^{-1}c - c\sqrt{1-c^2}\right]} \left[\begin{array}{l} \dfrac{x/l - d/l}{d/l} \ln \left| \dfrac{\sqrt{\dfrac{x/l}{1-x/l}} + \sqrt{\dfrac{d/l}{1-d/l}}}{\sqrt{\dfrac{x/l}{1-x/l}} - \sqrt{\dfrac{d/l}{1-d/l}}} \right| \\ -\dfrac{\sqrt{x/l(1-x/l)}}{d/l} \cos^{-1}\left(1 - \dfrac{2d}{l}\right) \end{array} \right] \right\} x > 0 \tag{14}$$

Fig. 3 displays the range of contact pressure distributions possible, as a function of d/l. As $d/l \to 1$ the Hertz semi-ellipse is recovered, whilst as d/l becomes small the effect of the edge radius becomes negligible, and the problem approximates that of a tilted square ended punch. The limit $d/l \to 0$ cannot easily be portrayed on this plot, but will be addressed fully when asymptotes are discussed in Section 4.

3.2. Shear tractions

Suppose, now, that a shearing force, Q, is gradually applied. Clearly, if $Q = \pm fP$, then $q(x) = \pm fp(x)$ everywhere. However, more interestingly, if the contact is not sliding, and a partial slip regime results the Ciavarella–Jäger theorem (Ciavarella, 1998; Jäger, 1998) may be employed, so that by scaling and shifting the sliding shearing traction distribution, to deduce the corrective shearing traction within the stick region, we arrive at the following solution for the shearing traction distribution (see Fig. 1, inset):

$$q(x) = \begin{cases} -fp(x), & 0 < x < d' \\ -fp(x) + q'(x), & d' < x < l - (d - d') - L \\ -fp(x), & l - (d - d') - L < x < l \end{cases} \tag{15}$$

where

$$q'(x) = \frac{f\beta(L/l)}{A\pi}\left\{\begin{array}{l} \dfrac{x-d}{d'}\ln\left|\dfrac{\sqrt{\dfrac{x-(d-d')}{L-(x-(d-d'))}}+\sqrt{\dfrac{d'}{L-d'}}}{\sqrt{\dfrac{x-(d-d')}{L-(x-(d-d'))}}-\sqrt{\dfrac{d'}{L-d'}}}\right| \\ -\dfrac{\sqrt{(x-(d-d'))[L-(x-(d-d'))]}}{d'}\cos^{-1}\left(1-\dfrac{2d'}{L}\right) \end{array}\right\} \tag{16}$$

with the extent of the stick zone, L, given by

$$L/l = \sqrt{1 - Q/fP} \tag{17}$$

and its position with respect to the reference point (d', Fig. 1) given by

$$d'/l = (1 - L/l)(d/l) \tag{18}$$

Fig. 4 shows the normalised shear tractions for $d/l = 0.2$ and various values of Q/fP.

3.3. Muskhelishvili potential

We turn, now, to a treatment of the internal state of stress. This is perhaps best represented in terms of a Muskhelishvili potential, from which the individual stress components can be found by standard means (Hills et al., 1993). The potential itself is found from the following integral evaluated along the surface

$$\Phi(z) = \frac{1}{2\pi i}\int_0^l \frac{p(t)}{t-z}\mathrm{d}t, \quad z = x + iy \tag{19}$$

Writing

$$z = \frac{l}{2}(\zeta + 1), \quad d = \frac{l}{2}(1 - c) \tag{20}$$

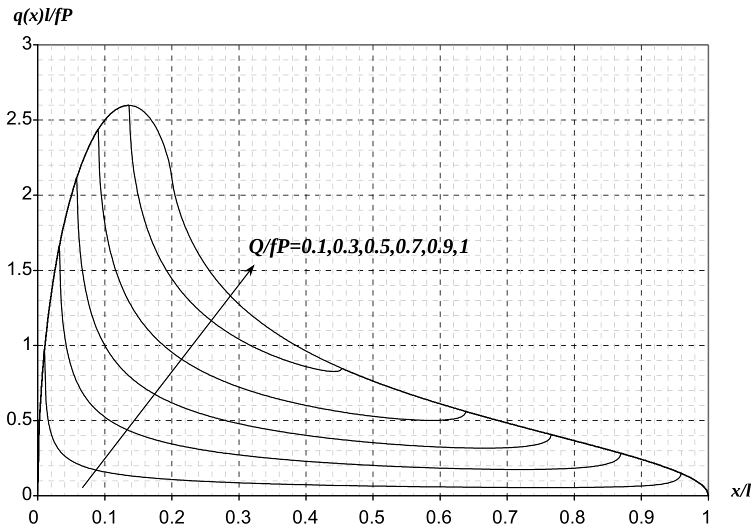

Fig. 4. Shear traction distributions ($d/l = 0.2$).

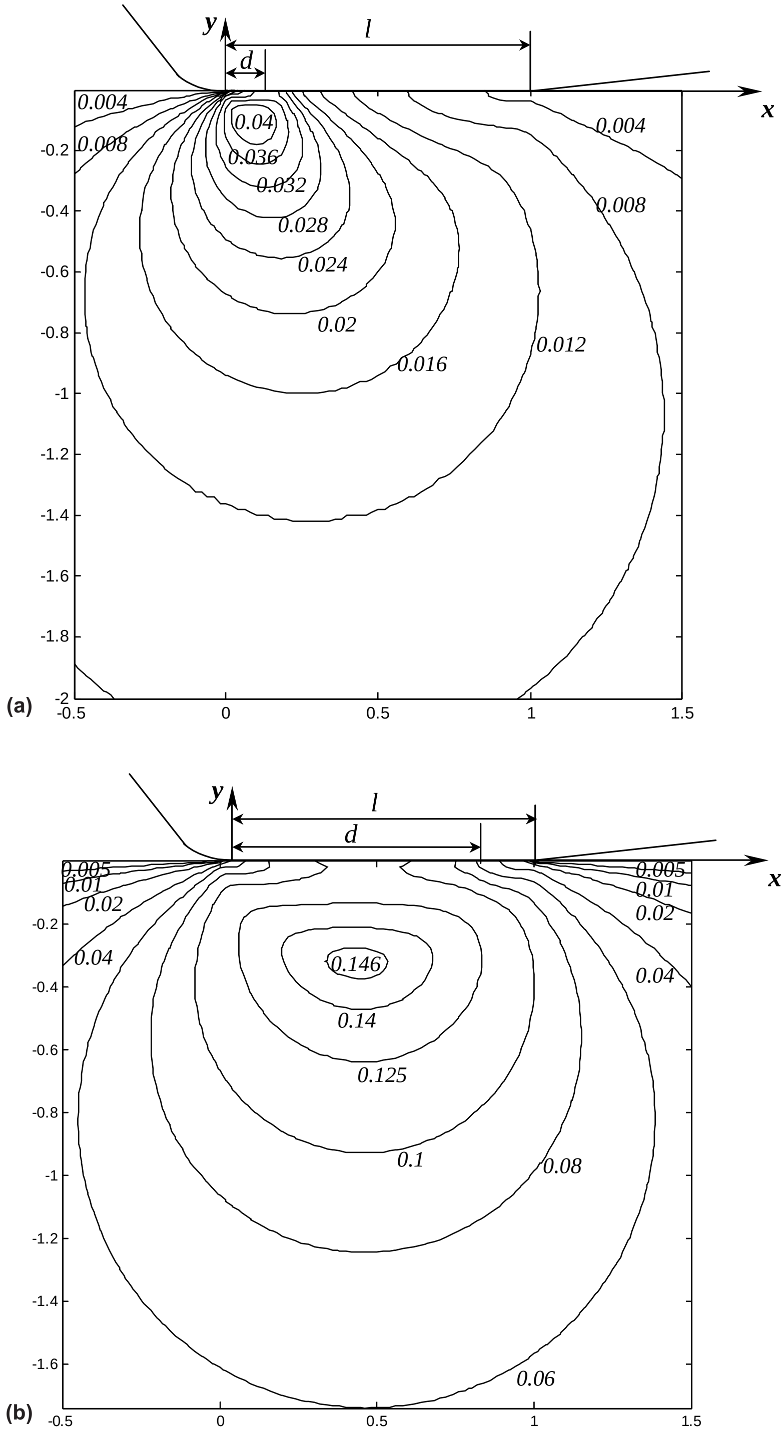

Fig. 5. Internal stresses for frictionless configurations: $\left(AR\sqrt{J_2}\right)/l$ for (a) $d/l = 0.2$ and (b) $d/l = 0.8$.

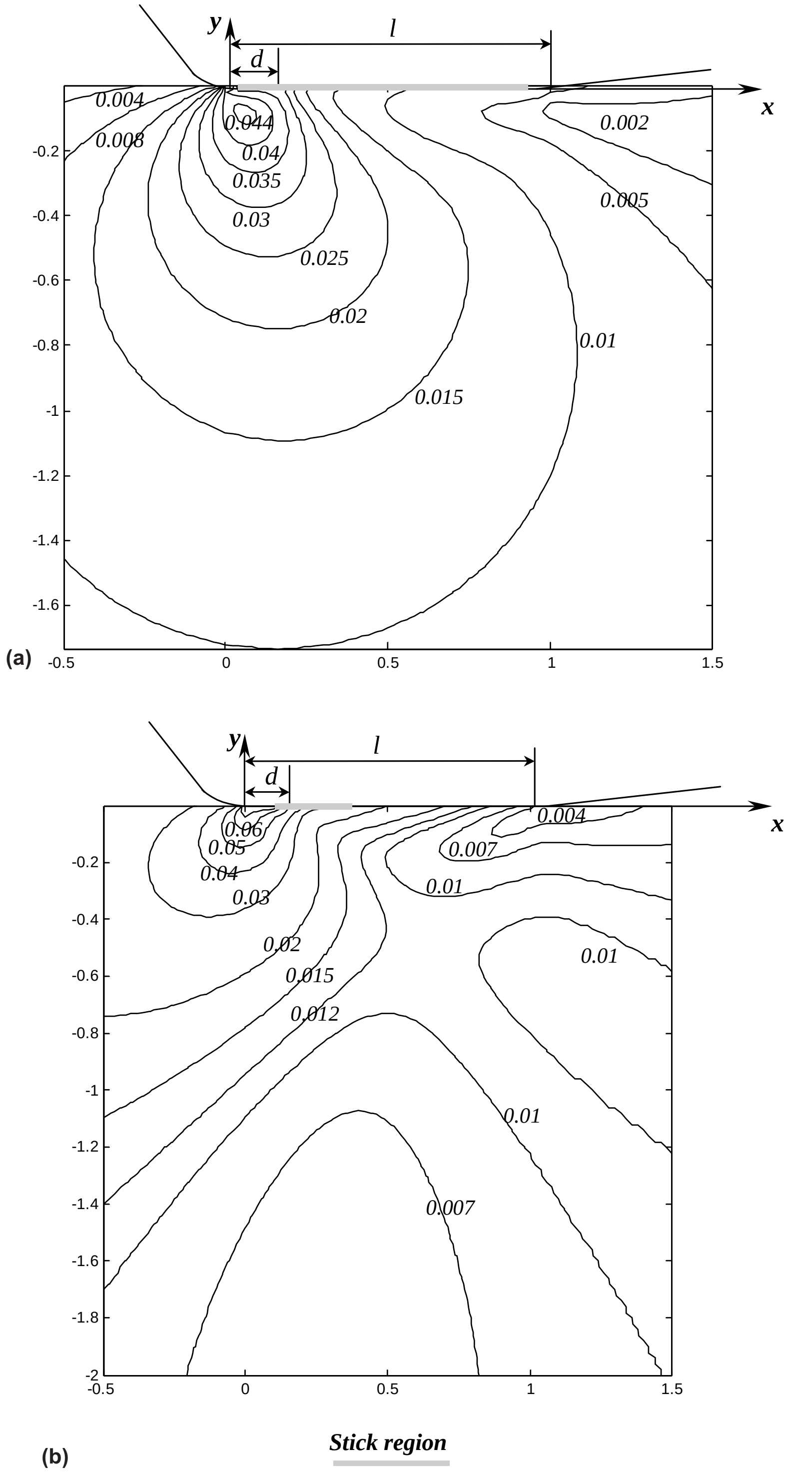

Fig. 6. Internal stresses for partial slip configurations: $\left(AR\sqrt{J_2}\right)/l$ for $d/l = 0.2$, $f = 0.5$ (a) $Q/fP = 0.2$ and (b) $Q/fP = 0.8$.

we find from Appendix B that

$$\Phi(z) = \frac{\beta}{2\pi \mathrm{i} A(1-c)} \left[\begin{array}{l} \sqrt{1-c^2} - \left(c + \sqrt{\zeta^2 - 1}\right)\cos^{-1}c \\ +2(\zeta + c)\tan^{-1}\left(\sqrt{\frac{1-c}{1+c}}\sqrt{\frac{\zeta-1}{\zeta+1}}\right) \end{array} \right] \tag{21}$$

We may ignore the constant terms. Now

$$\begin{aligned} &\zeta + 1 = \frac{2z}{l}, \quad \zeta - 1 = 2\left(\frac{z}{l} - 1\right), \quad \frac{\zeta - 1}{\zeta + 1} = \frac{z-l}{z}, \quad \zeta^2 - 1 = \frac{4}{l^2}z(z-l) \\ &1 + c = 2\left(1 - \frac{d}{l}\right), \quad 1 - c = \frac{2d}{l}, \quad \frac{1-c}{1+c} = \frac{d}{l-d}, \quad \zeta + c = \frac{2}{l}(z-d), \end{aligned} \tag{22}$$

so that

$$\Phi(z) = \frac{1}{2\pi \mathrm{i} AR}\left[2(z-d)\tan^{-1}\left(\sqrt{\frac{d}{l-d}}\sqrt{\frac{z-l}{z}}\right) - \sqrt{z(z-l)}\cos^{-1}\left(1 - \frac{2d}{l}\right)\right] \tag{23}$$

As a check, the pressure distribution has been successfully evaluated using (23) and applying the Plemelj formulae (see Appendix C). The closed form solution for partial slip contact configurations has also been derived by superposition using the results from Ciavarella (1998) and Jäger (1998) but it is not reported here for brevity.

Of course all stress components at any point may be found from this result, but perhaps the best way of displaying their significance is to look at contours of the von Mises parameter, normalised with respect to the mean load, i.e. $(AR\sqrt{J_2})/l$, and these are shown for representative values of d/l in Fig. 5, for frictionless contact. Space limitations preclude a detailed discussion of the behaviour when frictional tractions are present, but a representative solution is shown in Fig. 6, for partial slip configurations when $d/l = 0.2$, $f = 0.5$ and $Q/fP = 0.2, 0.8$.

4. Asymptotic forms

First, the question arises of the possible application of this solution to develop a second-order asymptote: we have already successfully derived an asymptotic solution for a second-order contact asymptote, in which one contacting body has the form of a semi-infinite quarter plane, having a local edge radius (Dini and Hills, 2003). This has been employed to obtain a refined solution to several 'nearly complete' contact problems having only a small edge radius, and it would clearly be very useful if an equivalent solution for a a tilted semi-infinite rounded punch might be found. However, it is found that, if the limit $l \to \infty$ is taken directly, keeping d finite, the value of α goes to zero, so that the non-tilted solution is again approached.[1]

Notwithstanding this limitation, it has nevertheless proved possible to fit asymptotes within *this* problem, and which, we will demonstrate, substantially recover the 'semi-infinite' solution required. Starting from the general equation for the contact pressure (Eq. (7)), we see that, if $x \ll d$ square root bounded behaviour is to be anticipated (Dini and Hills, 2004), and this is indeed revealed by expansion of the general result, to give

[1] Consider Eq. (7). As $l \to \infty$ (and therefore $c \to 1$) we get:

$$p(x) = \frac{\beta}{A\pi}\left\{\frac{x-d}{d}\ln\left|\frac{\sqrt{x}+\sqrt{d}}{\sqrt{x}-\sqrt{d}}\right| - 2\sqrt{\frac{x}{d}}\right\}, \quad \frac{\alpha}{\beta} \to 0.$$

$$p(x) = \frac{2\beta\sqrt{l}}{A\pi d}\cos^{-1}\left(1 - \frac{2d}{l}\right)\sqrt{x} \equiv K_b^N\sqrt{x}, \quad x \ll d \tag{24}$$

Further, if we let the radius of the punch become very small compared with the flat length of the contact ($R \ll l$) we would expect the solution to approximate that of a tilted square ended rigid punch pressed into an incompressible half-plane. Thus, if we bear in mind also that this form of the solution is valid only if the punch is rigid, we expect square root singular behaviour in the contact pressure, and indeed this is what is revealed

$$p(x) = \frac{\frac{2\beta\sqrt{l}}{3A\pi}\cos^{-1}\left(1 - \frac{2d}{l}\right)}{\sqrt{x}} \equiv \frac{K_s^N}{\sqrt{x}}, \quad x \gg d, x \ll l \text{ and } l \ggg d \tag{25}$$

This result is derived in Appendix D. It follows that, if the punch is made extremely wide, and $R \ll l$ we can find a range of values of x ($\ll l$), in which the behaviour of the contact pressure can nevertheless be thought of as having the form

$$p(x) = K_b^N\sqrt{x} + K_s^N/\sqrt{x} \tag{26}$$

(Fig. 7) and with an intermediate region where terms of order $O(x^{3/2}, x^{-3/2})$ are also significant. A comparison of Eqs. (24) and (25) reveals that

$$\frac{K_s^N}{K_b^N} = \frac{d}{3} \tag{27}$$

while the 'contact law', giving the value of d in terms of independent variables is

$$d = \left(\frac{3\pi ARK_s}{2}\right)^{2/3} \tag{28}$$

We now consider the effect of a monotonically increasing shearing force insufficient to cause body motion (sliding), applied to the asymptotic problem. It will be appreciated that the solution can be found

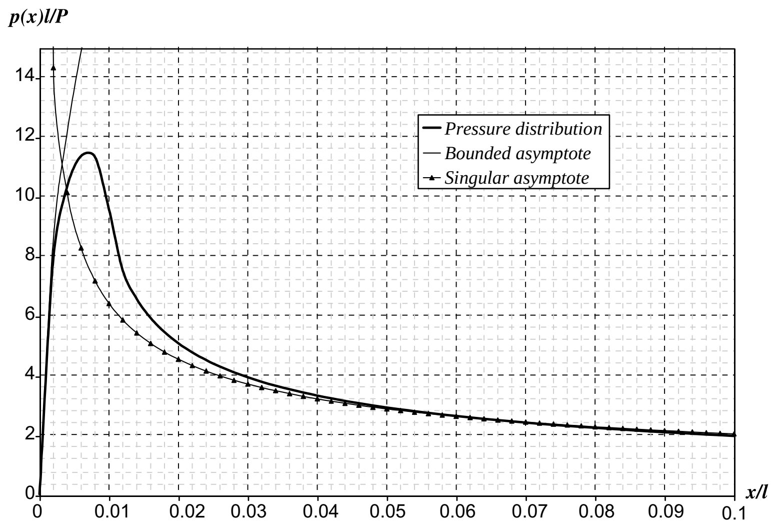

Fig. 7. Asymptotic expansions for the tilted punch pressure distribution at the edge of the contact: $d/l = 0.01$.

by a combination of scaling and shifting of the sliding solution, using the Ciavarella–Jäger theorem, as stated in Section 3.2 in relation to the corresponding finite problem. It follows that, when the process of taking the geometric limit $L/d \to \infty$ is applied to the shear traction distribution, the same trends are found which appeared for the contact pressure. Thus, the solution for the shear traction for the tilted, semi-infinite flat and rounded punch is the *same* as that for the untilted case, which was addressed fully in Dini and Hills (2003).

This set of results, employed together, may be used to serve in the same way as the 'flat' semi-infinite flat and rounded punch does, i.e. it may be used to add detail, viz. the influence of very local rounding, to a 'perfectly sharp' punch solution, because the ratio between the singular and bounded asymptotes, and also the contact law, have all been demonstrated to continue to apply in the tilted punch case.

4.1. Example problem

As it is always possible to fit conventional square root singular asymptotes to pressure and shear into the corner of a *tilted* punch, the influence of rounding may be added straightforwardly. In practice we would expect this procedure to be applied to a complex punch whose contact pressure had been found numerically but here, for the sake of analytical clarity, we will consider briefly a classical problem as an example: a rigid punch of half-width b subject to a load P applied at a distance t ($< b/2$ so as to maintain full contact) from the centreline pressed into an incompressible half-plane, so that it tilts. This gives rise to a contact pressure

$$p(x) = \frac{P}{\pi\sqrt{b^2 - x^2}}\left[1 + \frac{2tx}{b^2}\right]. \tag{29}$$

where, here, x is measured from the punch centreline. It follows that, as $x \to \pm b$ the contact pressure is square root singular, and, at the end $x \to b$ the multiplicative constant defining the contact pressure is given by

$$K_s^N = \frac{P}{\pi\sqrt{2b}}\sqrt{1 + \left(\frac{2t}{b}\right)} \tag{30}$$

Thus, if the titled punch in question was subject to slight rounding, of radius R, we would infer that, close to this edge

$$p(x) = \left(\frac{4P\sqrt{1 + \left(\frac{2t}{b}\right)}}{3\sqrt{2}\sqrt{b}(\pi AR)^2}\right)^{1/3}\sqrt{x} \tag{31}$$

and the solution for the effect of a small amount of rounding has been found without further calculation.

Consider, now, the effect of a monotonically increasing shear, Q. If the contact is fully adhered, the shearing traction distribution, $q(x)$, is given by

$$q(x) = \frac{Q}{\pi\sqrt{b^2 - x^2}} \tag{32}$$

regardless of whether the punch is tilted or not, and hence

$$K^T = \frac{Q}{\pi}\sqrt{\frac{2}{b}}. \tag{33}$$

Complete adhesion will certainly be maintained if $|Q|/fP < (1 + 2|t|/b)$ and this is what we shall assume. From Dini and Hills (2003) the size of the slip region, d', is given by

$$\frac{K^T}{K_s^N} = 1 - \left(\frac{d'}{d}\right)^{3/2} \tag{34}$$

and hence

$$d' = d\left(1 - \frac{Q}{P\left(1 + \frac{2t}{b}\right)}\right). \tag{35}$$

5. Conclusions

The problem of a punch having the form of a flat face but with rounded edges, pressed into a contacting half-plane, has been solved on the basis of uncoupled half-plane theory. Results for the contact law and contact pressure distribution have been found, together with the internal state of stress, whether frictionless or sliding, through the Muskhelishvili potential, all in closed form. The effects of a monotonically increasing shearing force, insufficient to cause sliding have been found, and a detailed consideration of the asymptotic behaviour undertaken. The last has revealed that, although it is not formally possible to produce a solution for the tilted semi-infinite punch in a conventional way, it is possible to form an independent relationship between the very near edge behaviour and the moderately near edge behaviour, which serves an equivalent function. This may be used to add rounding detail to the edge of any notionally sharp contact edge.

Further, it has been shown that the solution for shearing traction distribution for a tilted semi-infinite flat and rounded punch is the same as that for an untilted punch. The pair of asymptotic solutions has then been used to solve an example problem, *viz.* a slightly rounded tilted square-ended finite punch, in complete contact.

Acknowledgements

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Appendix A. Details of the inversion of the integral equation

Combining (5) and (6) gives an integral equation for the unknown $p(x)$:

$$\frac{1}{\pi}\int_0^l \frac{p(t)}{t-x}\mathrm{d}t = -\frac{1}{A}\begin{cases} \alpha + \beta\left(1 - \frac{x}{d}\right), & 0 \leqslant x \leqslant d \\ \alpha, & d \leqslant x \leqslant l \end{cases} \tag{36}$$

Most of the standard results for Singular Integral Equations (SIE) are usually couched in terms of integrals over the range $[-1, 1]$ so we transform to this range. Put

$$t = \frac{l}{2}(s+1) \Longleftrightarrow s = \frac{2t}{l} - 1 \tag{37}$$

$$x = \frac{l}{2}(w+1) \Longleftrightarrow w = \frac{2x}{l} - 1. \tag{38}$$

Then

t or x	s or w
0	-1
d	$-c$
l	1

where

$$c = 1 - \frac{2d}{l}, \quad |\, c \,| < 1 \tag{39}$$

Then

$$1 - \frac{x}{d} = -(1-c)(c+w), \tag{40}$$

$$t - x = \frac{l}{2}(s-w), \quad \frac{\mathrm{d}t}{\mathrm{d}s} = \frac{l}{2} \tag{41}$$

Then (36) becomes

$$\frac{1}{\pi}\int_{-1}^{1} \frac{P(s)}{s-w}\,\mathrm{d}s = \frac{G(w)}{A} \tag{42}$$

where

$$G(w) = -\begin{cases} \alpha - \beta(c+w)/(1-c), & -1 \leqslant w \leqslant -c \\ \alpha, & -c \leqslant w \leqslant 1 \end{cases} \tag{43}$$

The solution of (42) which is *bounded-bounded* [the statement of the particular problem here indicates that $p(0) = p(l) = 0$ and therefore $P(-1) = P(1) = 0$] is

$$P(w) = -\frac{\sqrt{1-w^2}}{\pi}\int_{-1}^{1} \frac{G(s)/A}{\sqrt{1-s^2}(s-w)}\,\mathrm{d}s \tag{44}$$

subject to the compatibility requirement

$$\int_{-1}^{1} \frac{G(s)/A}{\sqrt{1-s^2}}\,\mathrm{d}s \equiv 0 \tag{45}$$

This is evaluated explicitly for the problem to give:

$$\frac{\alpha}{\beta} = \frac{1}{\pi}\left\{\frac{c\cos^{-1}c - \sqrt{1-c^2}}{1-c}\right\} \tag{46}$$

which has the property that $\alpha/\beta \to 0$ as $l \to \infty$.

From (44)

$$P(w) = -\frac{\sqrt{1-w^2}}{A\pi}\int_{-1}^{1}\frac{G(s)}{\sqrt{1-s^2}(s-w)}\mathrm{d}s$$
$$= \frac{\sqrt{1-w^2}}{A\pi}\left\{\int_{-1}^{-c}\frac{\alpha-\beta(c+s)/(1-c)}{\sqrt{1-s^2}(s-w)}\mathrm{d}s + \int_{-c}^{1}\frac{\alpha}{\sqrt{1-s^2}(s-w}\mathrm{d}s\right\}$$
$$= \frac{\sqrt{1-w^2}}{A\pi}\left\{\int_{-1}^{1}\frac{\alpha}{\sqrt{1-s^2}(s-w)}\mathrm{d}s - \frac{\beta}{1-c}\int_{-1}^{-c}\frac{c+s}{\sqrt{1-s^2}(s-w)}\mathrm{d}s\right\}$$
$$= -\frac{\sqrt{1-w^2}\beta}{A\pi(1-c)}\int_{-1}^{-c}\frac{(c+s)\mathrm{d}s}{\sqrt{1-s^2}(s-w)} \tag{47}$$

as the first integral on the third line above is identically zero. So, using the substitution $s \to -s$ in the integral, we get

$$P(w) = \frac{\sqrt{1-w^2}\beta}{A\pi(1-c)}\int_{c}^{1}\frac{(c-s)\mathrm{d}s}{\sqrt{1-s^2}(s+w)}$$
$$= \frac{\sqrt{1-w^2}\beta}{A\pi(1-c)}\left\{(c+w)\int_{c}^{1}\frac{\mathrm{d}s}{\sqrt{1-s^2}(s+w)} - \int_{c}^{1}\frac{\mathrm{d}s}{\sqrt{1-s^2}}\right\}$$
$$= \frac{\sqrt{1-w^2}\beta}{A\pi(1-c)}\left\{(c+w)\int_{c}^{1}\frac{\mathrm{d}s}{\sqrt{1-s^2}(s+w)} - \cos^{-1}c\right\} \tag{48}$$

Remember that $|w| < 1$ so that this integral can be CPV (Cauchy Principal Value) or regular depending on the location of w, i.e. if $-1 < w < -c$ then the integral is CPV but if $c < w < 1$ then the integral is regular.

$$I = \int_{c}^{1}\frac{1}{\sqrt{1-s^2}(s+w)}\mathrm{d}s, \quad |w| < 1 \tag{49}$$

Put $s = \cos\theta$

$$I = \int_{0}^{\phi}\frac{\mathrm{d}\theta}{w+\cos\theta}, \quad \cos c = \phi \tag{50}$$

Put $u = \tan(\theta/2)$

$$I = \frac{2}{1-w}\int_{0}^{u_0}\frac{\mathrm{d}u}{k^2-u^2}, \quad k^2 = \frac{1+w}{1-w}, \quad u_0 = \tan\left(\frac{\phi}{2}\right) \tag{51}$$

This is a simple integral which has its CPV value equal to its regular value

$$\int_{0}^{u_0}\frac{\mathrm{d}u}{k^2-u^2} = \frac{1}{2k}\lim_{\varepsilon\to 0}\left\{\left[\ln\left|\frac{k+u}{k-u}\right|\right]_{0}^{k-\varepsilon} + \left[\ln\left|\frac{k+u}{k-u}\right|\right]_{k+\varepsilon}^{u_0}\right\}, \quad 0 < k < u_0$$
$$= \frac{1}{2k}\lim_{\varepsilon\to 0}\left[\ln\left|\frac{2k-\varepsilon}{\varepsilon}\,\frac{\varepsilon}{2k+\varepsilon}\,\frac{k+u_0}{k-u_0}\right|\right] = \frac{1}{2k}\ln\left|\frac{k+u_0}{k-u_0}\right|. \tag{52}$$

and this is also the value if $k > u_0$. So

$$I = \frac{1}{\sqrt{1-w^2}} \ln \left| \frac{\sqrt{\frac{1+w}{1-w}} + \sqrt{\frac{1-c}{1+c}}}{\sqrt{\frac{1+w}{1-w}} - \sqrt{\frac{1-c}{1+c}}} \right| \tag{53}$$

Hence

$$P(w) = \frac{\beta}{A\pi(1-c)} \left\{ (c+w) \ln \left| \frac{\sqrt{\frac{1+w}{1-w}} + \sqrt{\frac{1-c}{1+c}}}{\sqrt{\frac{1+w}{1-w}} - \sqrt{\frac{1-c}{1+c}}} \right| - \sqrt{1-w^2}\cos^{-1} c \right\} \tag{54}$$

Noting that

$$\begin{aligned} &c = 1 - \frac{2d}{l}, \quad \frac{1-c}{1+c} = \frac{d}{l-d}, \quad w = \frac{2x}{l} - 1, \quad \frac{1+w}{1-w} = \frac{x}{l-x}, \\ &1 - w^2 = \frac{4}{l^2} x(l-x), \quad c + w = \frac{2}{l}(x-d) \end{aligned} \tag{55}$$

we have

$$p(x) = \frac{\beta}{A\pi} \left\{ \frac{x-d}{d} \ln \left| \frac{\sqrt{\frac{x}{l-x}} + \sqrt{\frac{d}{l-d}}}{\sqrt{\frac{x}{l-x}} - \sqrt{\frac{d}{l-d}}} \right| - \frac{\sqrt{x(l-x)}}{d} \cos^{-1}\left(1 - \frac{2d}{l}\right) \right\} \tag{56}$$

A.1. Applied load P

The applied load is

$$P = \int_0^l p(x)\mathrm{d}x \tag{57}$$

Making the transformation to $[-1, 1]$ gives

$$\begin{aligned} P &= \frac{l}{2} \int_{-1}^{1} P(w)\mathrm{d}w = \frac{l}{2} \int_{-1}^{1} \left\{ -\frac{\sqrt{1-w^2}}{\pi} \int_{-1}^{1} \frac{G(s)/A}{\sqrt{1-s^2}(s-w)} \mathrm{d}s \right\} \mathrm{d}w, \\ &= -\frac{l}{2A\pi} \int_{-1}^{1} \frac{G(s)}{\sqrt{1-s^2}} \left\{ \int_{-1}^{1} \frac{\sqrt{1-w^2}}{s-w} \mathrm{d}w \right\} \mathrm{d}s = -\frac{l}{2A\pi} \int_{-1}^{1} \frac{G(s)}{\sqrt{1-s^2}} \{\pi s\} \mathrm{d}s. \end{aligned} \tag{58}$$

i.e.

$$P = -\frac{l}{2A} \int_{-1}^{1} \frac{sG(s)\mathrm{d}s}{\sqrt{1-s^2}} \tag{59}$$

Performing an integration by parts gives

$$P = -\frac{l}{2A} \left[-\sqrt{1-s^2}G(s) + \int \sqrt{1-s^2}G'(s)\mathrm{d}s \right]_{-1}^{1} \tag{60}$$

i.e.

$$P = -\frac{l}{2A}\int_{-1}^{1}\sqrt{1-s^2}G'(s)\mathrm{d}s \tag{61}$$

This may be evaluated for the case in question

$$G'(s) = \begin{cases} \beta/(1-c), & -1 \leqslant s \leqslant -c \\ 0, & -c \leqslant s \leqslant 1 \end{cases} \tag{62}$$

Note that differentiating G causes the twist dependence to vanish, so P is independent of the twist.

$$P = -\frac{l}{2A}\left\{\int_{-1}^{-c}\sqrt{1-s^2}\left[\frac{\beta}{1-c}\right]\mathrm{d}s + \int_{-c}^{1}\sqrt{1-s^2}[0]\mathrm{d}s\right\} = -\frac{l\beta}{2A1-c}\int_{-1}^{-c}\sqrt{1-s^2}\mathrm{d}s$$

$$= -\frac{l\beta}{2A(1-c)}\int_{c}^{1}\sqrt{1-s^2}\mathrm{d}s \quad \text{using } s\to -s = \frac{l\beta}{4A(1-c)}\left[c\sqrt{1-c^2}-\cos^{-1}c\right]. \tag{63}$$

Remember that

$$c = 1-\frac{2d}{l} \to l = \frac{2d}{1-c} \tag{64}$$

So

$$P = \frac{\mathrm{d}\beta}{2A}\left[\frac{c\sqrt{1-c^2}-\cos^{-1}c}{(1-c)^2}\right] \tag{65}$$

A.2. Applied moment

Taking moments about $x = 0$ we have

$$M = Ps = \int_0^l xp(x)\mathrm{d}x. \tag{66}$$

Making the transformation to $[-1, 1]$ gives

$$M = \frac{l^2}{4}\int_{-1}^{1}(w+1)P(w)\mathrm{d}w = \frac{lP}{2}+\frac{l^2}{4}\int_{-1}^{1}wP(w)\mathrm{d}w \tag{67}$$

Now

$$\begin{aligned}\int_{-1}^{1}wP(w)\mathrm{d}w &= \int_{-1}^{1}w\left\{-\frac{\sqrt{1-w^2}}{A\pi}\int_{-1}^{1}\frac{G(s)}{\sqrt{1-s^2}(s-w)}\mathrm{d}s\right\}\mathrm{d}w \\ &= \frac{1}{A\pi}\int_{-1}^{1}\frac{G(s)}{\sqrt{1-s^2}}\left\{\int_{-1}^{1}\frac{w\sqrt{1-w^2}}{w-s}\mathrm{d}w\right\}\mathrm{d}s \\ &= \frac{1}{A\pi}\int_{-1}^{1}\frac{G(s)}{\sqrt{1-s^2}}\left\{\int_{-1}^{1}\frac{w-w^3}{\sqrt{1-w^2}(w-s)}\mathrm{d}w\right\}\mathrm{d}s \\ &= \frac{1}{4A\pi}\int_{-1}^{1}\frac{G(s)}{\sqrt{1-s^2}}\left\{\int_{-1}^{1}\frac{T_1(w)-T_3(w)}{\sqrt{1-w^2}(w-s)}\mathrm{d}w\right\}\mathrm{d}s \\ &= \frac{1}{4A\pi}\int_{-1}^{1}\frac{G(s)}{\sqrt{1-s^2}}\{2\pi-4\pi s^2\}\mathrm{d}s \\ &= -\frac{1}{A}\int_{-1}^{1}\frac{s^2G(s)\mathrm{d}s}{\sqrt{1-s^2}} \quad \text{as} \int_{-1}^{1}\frac{G(s)\mathrm{d}s}{\sqrt{1-s^2}} = 0 \text{ (compatibility)}\end{aligned} \tag{68}$$

Re-using the compatibility equation we can write this as

$$\int_{-1}^{1} wP(w)\mathrm{d}w = \frac{1}{A}\int_{-1}^{1}\sqrt{1-s^2}G(s)\mathrm{d}s \tag{69}$$

Hence

$$M = \frac{lP}{2} + \frac{l^2}{4A}\int_{-1}^{1}\sqrt{1-s^2}G(s)\mathrm{d}s \tag{70}$$

Taking note of (61) we can also write this as

$$M = \frac{l^2}{4A}\int_{-1}^{1}\sqrt{1-s^2}(G(s)-G'(s))\mathrm{d}s \tag{71}$$

but the integral in (70) for our special case is given by

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-s^2}G(s)\mathrm{d}s &= \int_{-1}^{-c}\sqrt{1-s^2}\left[-\alpha+\frac{\beta(c+s)}{1-c}\right]\mathrm{d}s + \int_{-c}^{1}\sqrt{1-s^2}(-\alpha)\mathrm{d}s\\ &= -\alpha\int_{-1}^{1}\sqrt{1-s^2}\mathrm{d}s + \frac{\beta}{1-c}\int_{-1}^{-c}\sqrt{1-s^2}(c+s)\mathrm{d}s\\ &= -\alpha\int_{-1}^{1}\sqrt{1-s^2}\mathrm{d}s + \frac{\beta}{1-c}\int_{c}^{1}\sqrt{1-s^2}(c-s)\mathrm{d}s, \quad \text{using } s\to -s\\ &= -\frac{\alpha\pi}{2} + \frac{\beta}{1-c}\left[\frac{1}{2}c\cos^{-1}c - \frac{c^2}{6}\sqrt{(1-c^2)} - \frac{1}{3}\sqrt{(1-c^2)}\right]\end{aligned} \tag{72}$$

and

$$\alpha\pi = \frac{\beta}{1-c}\left[c\cos^{-1}c - \sqrt{1-c^2}\right]. \tag{73}$$

Hence

$$\int_{-1}^{1}\sqrt{1-s^2}G(s)\mathrm{d}s = \frac{\beta}{1-c}\left[\frac{1}{6}\sqrt{(1-c^2)} - \frac{c^2}{6}\sqrt{(1-c^2)}\right] = \frac{\beta(1-c^2)^{3/2}}{6(1-c)} \tag{74}$$

$$M = \frac{lP}{2} + \frac{l^2}{4A}\frac{\beta(1-c^2)^{3/2}}{6(1-c)} \tag{75}$$

Substituting (65) in (75) we have

$$M = \frac{\beta l^2}{24A}\left[\frac{(1-c^2)^{3/2} + 3c\sqrt{1-c^2} - 3\cos^{-1}c}{1-c}\right] \tag{76}$$

Appendix B. Muskhelishvili's potential

The Muskhelishvili potential is

$$\Phi(z) = \frac{1}{2\pi \mathrm{i}}\int_{0}^{l}\frac{p(t)}{t-z}\mathrm{d}t, \quad z = x+\mathrm{i}y \tag{77}$$

Make the substitution

$$z=\frac{l}{2}(\zeta+1),\quad t=\frac{l}{2}(s+1) \tag{78}$$

in (77) in order to standardise the interval

$$\Phi(z)=\frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}\frac{P(s)}{s-\zeta}\mathrm{d}s \tag{79}$$

and hence

$$P(s)=\frac{\sqrt{1-s^2}\beta}{A\pi(1-c)}\int_{c}^{1}\frac{(c-u)\mathrm{d}u}{\sqrt{1-u^2}(u+s)} \tag{80}$$

where $c = 1{-}2d/l$. Substituting (80) into (79) gives

$$\Phi(z)=\frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}\frac{1}{s-\zeta}\left\{\frac{\sqrt{1-s^2}\beta}{A\pi(1-c)}\int_{c}^{1}\frac{(c-u)\mathrm{d}u}{\sqrt{1-u^2}(u+s)}\right\}\mathrm{d}s \tag{81}$$

Interchanging the order of integration

$$\Phi(z)=\frac{\beta}{2\pi^2\mathrm{i}A(1-c)}\int_{c}^{1}\frac{(c-u)}{\sqrt{1-u^2}}\left\{\int_{-1}^{1}\frac{\sqrt{1-s^2}\mathrm{d}s}{(s-\zeta)(u+s)}\right\}\mathrm{d}u \tag{82}$$

Note that the inner integral is CPV, also note that

$$\frac{1}{(s-\zeta)(u+s)}\equiv\frac{1}{u+\zeta}\left(\frac{1}{s-\zeta}-\frac{1}{s+u}\right) \tag{83}$$

Hence the inner integral reads

$$\int_{-1}^{1}\frac{\sqrt{1-s^2}\mathrm{d}s}{(s-\zeta)(u+s)}=\frac{1}{u+\zeta}\left(\int_{-1}^{1}\frac{\sqrt{1-s^2}\mathrm{d}s}{s-\zeta}-\int_{-1}^{1}\frac{\sqrt{1-s^2}\mathrm{d}s}{s+u}\right) \tag{84}$$

Now the first integral on the RHS is regular (ζ is a complex number, but s is real) whilst the second is CPV. They are both easily evaluated using Gladwell (1980) to give

$$\int_{-1}^{1}\frac{\sqrt{1-s^2}\mathrm{d}s}{(s-\zeta)(u+s)}=\frac{\pi}{u+\zeta}\left(\sqrt{\zeta^2-1}-\zeta-\pi u\right) \tag{85}$$

Thus

$$\Phi(z)=\frac{\beta}{2\pi^2\mathrm{i}A(1-c)}\int_{c}^{1}\frac{(c-u)}{\sqrt{1-u^2}}\left\{\frac{\pi}{u+\zeta}\left(\sqrt{\zeta^2-1}-\zeta-u\right)\right\}\mathrm{d}u \tag{86}$$

The integrals required here give

$$\int_{c}^{1}\frac{(c-u)\mathrm{d}u}{\sqrt{1-u^2}(u+\zeta)}=\frac{2(c+\zeta)}{\sqrt{\zeta^2-1}}\tan^{-1}\left(\sqrt{\frac{1-c}{1+c}}\sqrt{\frac{\zeta-1}{\zeta+1}}\right)-\cos^{-1}c \tag{87}$$

$$\int_{c}^{1}\frac{u(c-u)\mathrm{d}u}{\sqrt{1-u^2}(u+\zeta)}=-\sqrt{1-c^2}+(\zeta+c)\cos^{-1}c-\frac{2\zeta(\zeta+c)}{\sqrt{\zeta^2-1}}\tan^{-1}\left(\sqrt{\frac{1-c}{1+c}}\sqrt{\frac{\zeta-1}{\zeta+1}}\right) \tag{88}$$

Thus

$$\Phi(z) = \frac{\beta}{2\pi\mathrm{i}A(1-c)}\left[\begin{array}{c}\sqrt{1-c^2} - \left(c + \sqrt{\zeta^2 - 1}\right)\cos^{-1}c \\ +2(\zeta + c)\tan^{-1}\left(\sqrt{\frac{1-c}{1+c}}\sqrt{\frac{\zeta-1}{\zeta+1}}\right)\end{array}\right] \tag{89}$$

Appendix C. Validation of the theory using the Plemelj formulae

As a check we may use the Plemelj formulae to re-find $p(x)$ from $\Phi(z)$. If

$$\Phi(z) = \frac{1}{2\pi\mathrm{i}}\int_0^l \frac{p(t)}{t-z}\mathrm{d}t, \quad z = x + \mathrm{i}y \tag{90}$$

Then, when z lies on the line of integration

$$\Phi^{\pm}(x) = \pm\frac{1}{2}p(x) + \frac{1}{2\pi\mathrm{i}}\int_0^l \frac{p(t)}{t-x}\mathrm{d}t, \quad 0 < x < l \tag{91}$$

which are the Plemelj formulae. So

$$p(x) = \Phi^+(x) - \Phi^-(x) \tag{92}$$

Now

$$\Phi(z) = \frac{\beta}{2\pi\mathrm{i}A(1-c)}\left[2(\zeta + c)\tan^{-1}\left(\sqrt{\frac{1-c}{1+c}}\sqrt{\frac{\zeta-1}{\zeta+1}}\right) - \sqrt{\zeta^2 - 1}\cos^{-1}c\right] \tag{93}$$

Let

$$w = \mathrm{Re}\,\zeta \tag{94}$$

Then

$$\left(\sqrt{\zeta - 1}\right)^+ = \mathrm{i}\sqrt{1-w}, \quad \left(\sqrt{\zeta - 1}\right)^- = \mathrm{i}\sqrt{1-w}, \tag{95}$$

$$\left(\sqrt{\zeta + 1}\right)^+ = \sqrt{1+w}, \quad \left(\sqrt{\zeta + 1}\right)^- = -\sqrt{1+w} \tag{96}$$

So

$$\Phi^+(x) - \Phi^-(x) = \frac{\beta}{2\pi\mathrm{i}A(1-c)}\left[\begin{array}{c}4(w + c)\tan^{-1}\left(\mathrm{i}\sqrt{\frac{1-c}{1+c}}\sqrt{\frac{1-w}{1+w}}\right) \\ -2\mathrm{i}\sqrt{1-w^2}\cos^{-1}c\end{array}\right] \tag{97}$$

and

$$\tan^{-1}(\mathrm{i}x) = \frac{\mathrm{i}}{2}\ln\left|\frac{1+x}{1-x}\right| \tag{98}$$

Therefore

$$p(x) = \frac{\beta}{\pi A(1-c)}\left[(w + c)\ln\left|\frac{1 + \sqrt{\frac{1-c}{1+c}}\sqrt{\frac{1-w}{1+w}}}{1 - \sqrt{\frac{1-c}{1+c}}\sqrt{\frac{1-w}{1+w}}}\right| - \sqrt{1-w^2}\cos^{-1}c\right] \tag{99}$$

Now

$$w+1=\frac{2x}{l},\quad w-1=2\left(\frac{x}{l}-1\right),\quad \frac{1-w}{1+w}=\frac{l-x}{x},\quad 1-w^2=\frac{4}{l^2}x(l-x)$$
$$1+c=2\left(1-\frac{d}{l}\right),\quad 1-c=\frac{2d}{l},\quad \frac{1-c}{1+c}=\frac{d}{l-d},\quad w+c=\frac{2}{l}(x-d), \tag{100}$$

So

$$p(x)=\frac{\beta}{\pi A d}\left[(x-d)\ln\left|\frac{1+\sqrt{\frac{d}{l-d}}\sqrt{\frac{l-x}{x}}}{1-\sqrt{\frac{d}{l-d}}\sqrt{\frac{l-x}{x}}}\right|-\sqrt{x(l-x)}\cos^{-1}c\right] \tag{101}$$

This agrees with the $p(x)$ already found (7).

Appendix D. Asymptotic expansion

Let us consider the pressure distribution

$$p(x)=P(w)=\frac{\sqrt{1-w^2}\beta}{A\pi(1-c)}\int_{-1}^{-c}\frac{(c+s)\mathrm{d}s}{\sqrt{1-s^2}(s-w)} \tag{102}$$

Reverting back to the physical parameters

$$t=\frac{l}{2}(s+1)\Longleftrightarrow s=\frac{2t}{l}-1 \tag{103}$$

$$x=\frac{l}{2}(w+1)\Longleftrightarrow w=\frac{2x}{l}-1. \tag{104}$$

Then

t or x	s or w
0	-1
d	$-c$
l	1

where

$$c=1-\frac{2d}{l},\quad |c|<1 \tag{105}$$

We have

$$p(x)=\frac{\beta}{\pi A d}\sqrt{x(l-x)}\int_0^d\frac{(d-t)}{\sqrt{t(l-t)}(x-t)}\mathrm{d}t \tag{106}$$

We are interested in the region $d\ll x<l$. So, as $0<t<d$, we have always that $t<x$. Hence the integral is regular and we may put

$$\frac{1}{x-t}=\frac{1}{x}\left(1-\frac{t}{x}\right)^{-1}=\frac{1}{x}\sum_0^\infty\left(\frac{t}{x}\right)^n \tag{107}$$

Therefore

$$p(x) = \frac{\beta}{\pi A d}\sqrt{\frac{l-x}{x}}\int_0^d \frac{(d-t)\sum_0^{\infty}(t/x)^n}{\sqrt{t(l-t)}}\mathrm{d}t \tag{108}$$

Thus, retaining the first term only in the summation (because $x >> t$)

$$p(x) \simeq \frac{\beta}{\pi A d}\sqrt{\frac{l-x}{x}}\int_0^d \frac{(d-t)}{\sqrt{t(l-t)}}\mathrm{d}t \tag{109}$$

Now

$$\frac{d}{\mathrm{d}t}\left\{(2d-l)\sin^{-1}\sqrt{\frac{t}{l}}+\sqrt{t(l-t)}\right\} = \frac{(d-t)}{\sqrt{t(l-t)}} \tag{110}$$

Therefore

$$\begin{aligned} p(x) &\simeq \frac{\beta}{\pi A d}\sqrt{\frac{l-x}{x}}\left[(2d-l)\sin^{-1}\sqrt{\frac{t}{l}}+\sqrt{t(l-t)}\right]_0^d \\ &= \frac{\beta}{\pi A d}\sqrt{\frac{l-x}{x}}\left[\sqrt{d(l-d)}-(l-2d)\sin^{-1}\sqrt{\frac{d}{l}}\right] \end{aligned} \tag{111}$$

Note that

$$1-c^2 = \frac{4d(l-d)}{l^2}, \quad \sin^{-1}\sqrt{\frac{d}{l}} = \frac{1}{2}\cos^{-1}c \tag{112}$$

Therefore also

$$p(x) \simeq \frac{\beta}{\pi A d}\frac{l}{2}\sqrt{\frac{l-x}{x}}\left[\sqrt{1-c^2}-c\cos^{-1}c\right] \tag{113}$$

And finally, since $x < l$ we have

$$\sqrt{l-x} = \sqrt{l}\left(1-\frac{x}{l}\right)^{1/2} \simeq \sqrt{l} \tag{114}$$

So

$$p(x) \simeq \frac{\beta l^{3/2}}{2\pi A d}\left[\sqrt{1-c^2}-c\cos^{-1}c\right]\frac{1}{\sqrt{x}} \tag{115}$$

which, for $d \ll l$ ($c \to 1$) gives Eq. (25).

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